CHAPTER

8

MODERN
CONTROL
DESIGN

The classical design techniques of Chapters 6 and 7 are based on the root-locus and frequency response that utilize only the plant output for feedback with a dynamic controller. In this final chapter on design, we employ modern control designs that require the use of all state variables to form a linear static controller. Modern control design is especially useful in multivariable systems; however, in this chapter the ideas of state-space design are illustrated using single-input, single-output systems.

One approach in modern control systems accomplished by the use of state feedback is known as *pole-placement design*. The pole-placement design allows all roots of the system characteristic equation to be placed in desired locations. This results in a regulator with constant gain vector $\mathbf{K}$.

The state-variable feedback concept requires that all states be accessible in a physical system, but for most systems this requirement is not met; i.e., some of the states are inaccessible. For systems in which all states are not available for feedback, a state estimator (observer) may be designed to implement the pole-placement design.

The other approach to the design of regulator systems is the optimal control problem where a specified mathematical performance criterion is minimized.

8.1 Pole-Placement Design

The control is achieved by feeding back the state variables through a regulator with constant gains. Consider the control system presented in the state-variable form
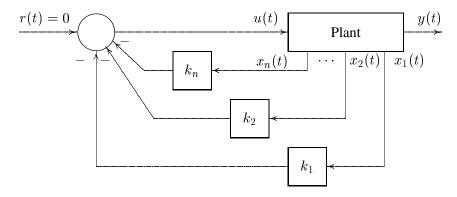
$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{8.1}$$

$$y(t) = \mathbf{C}\mathbf{x}(t)$$

Consider the block diagram of the system shown in Figure 8.1 with the following state feedback control

$$u(t) = -\mathbf{K}\mathbf{x}(t) \tag{8.2}$$

where $\mathbf{K}$ is a $1 \times n$ vector of constant feedback gains. The control system input $r(t)$ is assumed to be zero. The purpose of this system is to return all state variables to values of zero when the states have been perturbed.

FIGURE 8.1
Control system design via pole placement.

Substituting (8.2) into (8.1), the closed-loop system state-variable representation is

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) = \mathbf{A}_f\mathbf{x}(t) \tag{8.3}$$

The closed-loop system characteristic equation is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = 0 \tag{8.4}$$

Assume the system is represented in the phase variable canonical form as follows

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & & & & \\
0 & 0 & 0 & \ldots & 1 \\
-a_0 & -a_1 & -a_2 & \ldots & -a_{n-1}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}
+
\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)
\tag{8.5}
$$

Substituting for $\mathbf{A}$ and $\mathbf{B}$ into (8.4), the closed-loop characteristic equation for the control system is found

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = s^n + (a_{n-1} + k_n)s^{n-1} + \cdots + (a_1 + k_2)s + (a_0 + k_1) = 0 \quad (8.6)$$

For the specified closed-loop pole locations $-\lambda_1, \ldots, -\lambda_n$, the desired characteristic equation is

$$\alpha_c(s) = (s + \lambda_1) \cdots (s + \lambda_n) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0 \quad (8.7)$$

The design objective is to find the gain matrix $\mathbf{K}$ such that the characteristic equation for the controlled system is identical to the desired characteristic equation. Thus, the gain vector $\mathbf{K}$ is obtained by equating coefficients of equations (8.6) and (8.7).

$$k_i = \alpha_{i-1} - a_{i-1} \quad (8.8)$$

If the state model is not in the phase-variable canonical form, we can use the transformation technique of Chapter 3 Section 3.5 to transform the given state model to the phase-variable canonical form. The gain factor is obtained for this model and then transformed back to confirm with the original model. This procedure results in the following formula, known as *Ackermann's formula*.

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{S}^{-1} \alpha_c(\mathbf{A}) \quad (8.9)$$

where the matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} B & AB & A^2 B & \ldots & A^{n-1} B \end{bmatrix} \quad (8.10)$$

and the notation $\alpha_c(\mathbf{A})$ is given by

$$\alpha_c(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \cdots + \alpha_1 \mathbf{A} + \alpha_0 \mathbf{I} \quad (8.11)$$

The function $[\mathbf{K}, A_f] = \textbf{placepol(A, B, C, p)}$ is developed for the pole-placement design. $\mathbf{A, B, C}$ are system matrices and $\mathbf{p}$ is a row vector containing the desired closed-loop poles. This function returns the gain vector $\mathbf{K}$ and the closed-loop system matrix A_f. Also, the *MATLAB Control System Toolbox* contains two functions for pole-placement design. Function $\mathbf{K} = \textbf{acker(A, B, p)}$ is for single input systems, and function $\mathbf{K} = \textbf{place(A, B, p)}$, which uses a more reliable algorithm, is for multi-input systems.

The condition that must exist to place the closed-loop poles at the desired location is to be able to transform the given state model into phase-variable canonical form. That is, the controllability matrix $\mathbf{S}$, given in (8.10), must have a nonzero determinant. This characteristic is known as *controllability*.

Example 8.1

For the plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

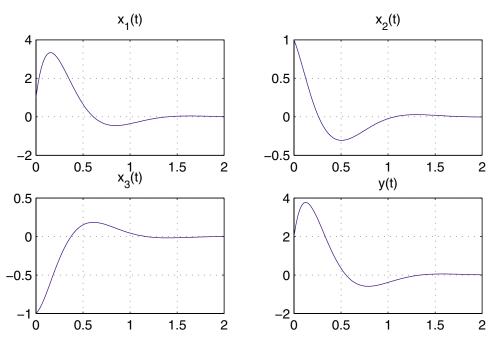$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \mathbf{x}$$

design a state feedback control to place the closed-loop pole at $-3 \pm j4$ and -8.
Obtain the initial condition response, given $x_1(0) = 1$, $x_2(0) = 1$ and $x_3(0) = -1$.
The following commands

```
A=[-1 0 0; -1 -2 0; 1 0 0];
B=[1; 0; 0];
C=[1 1 0 ]; D=0;
j=sqrt(-1);
P=[-3+j*4 -3-j*4  -8];              % desired closed-loop poles
[K,Af]=placepol(A,B,C,P);       % returns gain K and closed-loop
                                            % system matrix
% initial condition response
t=0:.02:2;
r=zeros(1,length(t));            % generates a row of zero input
x0=[1 1 -1];                                 % initial state
[y,x]=lsim(Af, B, C, D, r, t, x0);     % initial state response
subplot(2,2,1), plot(t, x(:,1)),title('x_1(t)'), grid
subplot(2,2,2), plot(t, x(:,2)),title('x_2(t)'), grid
subplot(2,2,3), plot(t, x(:,3)),title('x_3(t)'), grid
subplot(2,2,4), plot(t, y),title('y(t)'), grid
subplot(111)
```

result in

```
    Feedback gain vector K
      11    51    100

    Uncompensated Plant
    Transfer function:
         s^2 + s
    -----------------
    s^3 + 3 s^2 + 2 s

    Compensated system closed-loop
    Transfer function:
           s^2 + s
    -------------------------
    s^3 + 14 s^2 + 73 s + 200

    Compensated system matrix A - B*K
       -12    -51   -100
        -1     -2      0
```

The results are given in Figure 8.2.

FIGURE 8.2
Initial condition response for Example 8.1.

Example 8.2

A single-input single-output plant has the transfer function

$$G(s) = \frac{4}{s(s+1)(s+4)}$$

Obtain a state model for the plant and design a state feedback that will place the closed-loop pole at $-2 \pm j2$ and -5. Also, obtain the closed-loop transfer function for the controlled system.

The following commands

```
num=4; den=[1 5 4 0];
[A, B, C, D]=tf2ss(num, den)    % converts transfer function
                                %  to state model
j=sqrt(-1);
P=[-2+j*2 -2-j*2  -5];          % desired closed-loop poles
[K,Af]=placepol(A,B,C,P);    % returns gain K & closed-loop
                             % system matrix
```

result in

```
    A =
         -5     -4      0
          1      0      0
```

```
          0      1      0
    B =
          1
          0
          0
    C =
          0      0      4
    D =
          0
    Feedback gain vector K
        4    24    40
    Uncompensated Plant
    Transfer function:
             4
    -----------------
    s^3 + 5 s^2 + 4 s
    Compensated system closed-loop
    Transfer function:
               4
    -----------------------
    s^3 + 9 s^2 + 28 s + 40
    Compensated system matrix A - B*K
    -9    -28    -40
     1      0      0
     0      1      0
```

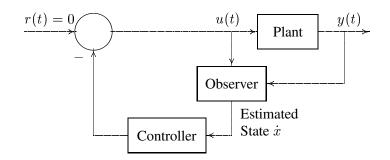From the above results the closed-loop transfer function for the controlled plant is

$$T(s) = \frac{C(s)}{R(s)} = \frac{4}{s^3 + 9s^2 + 28s + 40}$$

8.2 Controllability

A system is said to be controllable when the plant input u can be used to transfer the plant from any initial state to any arbitrary state in a finite time. The plant described by (8.1) with the system matrix having dimension $n \times n$ is completely state controllable if and only if the controllability matrix **S** in (8.10) has a rank of n. The function **S=cntrable(A, B)** is developed which returns the controllability matrix **S** and determines whether or not the system is state controllable.

8.3 Observer Design

In the pole-placement approach to the design of control systems, it was assumed that all state variables are available for feedback. However, in practice it is impractical to install all the transducers which would be necessary to measure all of the states. If the state variables are not available because of system configuration or cost, an observer

FIGURE 8.3
State feedback design with an observer.

or estimator may be necessary. The observer is an estimator algorithm based on the mathematical model of the system. The observer creates an estimate $\hat{\mathbf{x}}(t)$ of the states from the measurements of the output $y(t)$. The estimated states, rather than the actual states, are then fed to the controller. One scheme is shown in Figure 8.3.

Consider a system represented by the state and output equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{8.12}$$

$$y(t) = \mathbf{C}\mathbf{x}(t) \tag{8.13}$$

Assume that the state $\mathbf{x}(t)$ is to be approximated by the state $\hat{\mathbf{x}}(t)$ of the dynamic model

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{G}(y(t) - \hat{y}(t)) \tag{8.14}$$

$$\hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t) \tag{8.15}$$

Subtracting (8.14) from (8.12), and (8.15) from (8.13), we have

$$(\dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)) = \mathbf{A}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) - \mathbf{G}(y(t) - \hat{y}(t)) \tag{8.16}$$

$$(y(t) - \hat{y}(t)) = \mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) \tag{8.17}$$

where $\mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is the error between the actual state vector and the estimated vector, and $y(t) - \hat{y}(t)$ is the error between the actual output and the estimated output. Substituting the output equation into the state equation, we obtain the equation for the error between the estimated state vector and the actual state vector.

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{G}\mathbf{C})\mathbf{e}(t) = \mathbf{A}_{\mathbf{e}}\mathbf{e}(t) \tag{8.18}$$

where

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t) \tag{8.19}$$

If $\mathbf{G}$ is chosen such that eigenvalues of matrix $\mathbf{A} - \mathbf{G}\mathbf{C}$ all have negative real parts, then the steady-state value of the estimated state vector error $\mathbf{e}(t)$ for any initial condition will tend to zero. That is, $\hat{\mathbf{x}}(t)$ will converge to $\mathbf{x}(t)$. The design of the observer is similar to the design of the controller. However, the eigenvalues of $\mathbf{A} - \mathbf{G}\mathbf{C}$ must

be selected to the left of the eigenvalues of **A**. This ensures that the observer dynamic is faster than the controller dynamic for providing a rapid updated estimate of the state vector.

The estimator characteristic equation is given by

$$|s\mathbf{I} - \mathbf{A} + \mathbf{GC}| = 0 \tag{8.20}$$

For a specified speed of response, the desired characteristic equation for the estimator is

$$\alpha_e(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0 = 0 \tag{8.21}$$

Thus, the estimator gain **G** is obtained by equating coefficients of (8.20) and (8.21). This is identical to the pole-placement technique, and **G** is found by the application of Ackermann's formula

$$\mathbf{G} = \alpha_e(\mathbf{A}) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{8.22}$$

and the notation $\alpha_e(\mathbf{A})$ is given by

$$\alpha_e(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \cdots + \alpha_1\mathbf{A} + \alpha_0\mathbf{I} \tag{8.23}$$

The function **[G, A_e] = observer(A, B, C, p_e)** is developed for the estimator. p_e is the desired estimator eigenvalues. This function returns the gain vector **G** and the closed-loop system matrix **A** $_f$.

The necessary condition for the design of an observer is that all the states can be observed from the measurements of the output. This characteristic is known as *observability*.

Example 8.3

For the plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 16 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}$$

design a full-state observer such that the observer is critically damped with eigenvalues at -8 and -8.

The following commands

```
A=[ 0 1; 16  0];
B=[0; 1];
C=[1 0];  D=0;
Pe=[-8 -8];                    % desired observer eigenvalues
[K,Ae]= observer(A,B,C,Pe); % returns gain K and closed-loop
```

result in

```
Estimator gain vector G
   16
   80

Open-loop Plant
Transfer function:
    1
 --------
 s^2 - 16
Error matrix A - G*C
     -16      1
     -64      0
```

8.4   Observability

A system is said to be observable if the initial vector $\mathbf{x}(t)$ can be found from the measurement of $u(t)$ and $y(t)$. The plant described by (8.12) is completely state observable if the inverse matrix in (8.22) exists. The function **V=obsvable(A, C)** returns the observability matrix **V** and determines whether or not the system is state observable.

8.5   Combined Controller-Observer Design

Consider the system represented by the state and output equations (8.12) and (8.13) with the state feedback control based on the observed state $\hat{\mathbf{x}}(\mathbf{t})$ given by

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t) \tag{8.24}$$

Substituting in (8.12), the state equation becomes

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) + \mathbf{BK}\mathbf{e}(t) \tag{8.25}$$

Combining the above equation with the error equation given by (8.19), we have

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{GC} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{bmatrix} \tag{8.26}$$

The function **[K, G, A_c] = placeobs(A, B, C, p, p_e)** is developed for the combined controller-observer design. **A** is the system matrix, **B** is the input column vector, and **C** is the output row vector. **p** is a row vector containing the desired closed-loop poles and p_e is the desired observer eigenvalues. The function displays the gain vectors **K** and **G**, open-loop plant transfer function, and the controlled system closed-loop transfer function. Also, the function returns the gain vector **K**, and the combined system matrix in (8.26).

Example 8.4

For the system of Example 8.3, design a controller-observer system such that the desired closed-loop poles for the system are at $-1 \pm j2$. Choose the eigenvalues of the observer gain matrix to be $p_{e_1} = p_{e_2} = -8$.

The following commands

```
A=[ 0 1; 16  0];
B=[0; 1];
C=[1 0];  D=0;
j=sqrt(-1);
P=[-1+j*2 -1-j*2];                  % desired regulator roots
Pe=[-8 -8];                         % desired observer roots
[K,G,Af]= placeobs(A,B,C,P,Pe); % returns gain K,G & closed-loop
                                    % system matrix
```

result in

```
    Feedback gain vector K
      21   2
    Estimator gain vector G:
      16
      80
    Open-loop Plant
    Transfer function:
       1
    --------
    s^2 - 16
    Controller-estimator
    Transfer function:
      496 s + 2192
    ----------------
    s^2 + 18 s + 117
    Controlled system closed-loop
    Transfer function:
            496 s + 2192
    ------------------------------------
    s^4 + 18 s^3 + 101 s^2 + 208 s + 320
    Combined controller observer system matrix
     0     1     0     0
    -5    -2    21     2
     0     0   -16     1
     0     0   -64     0
```

From the above results the controller-observer transfer function is

$$G_{ce} = \frac{496s + 2192}{s^2 + 18s + 117}$$

The closed-loop transfer function for the controlled plant is

$$T(s) = \frac{C(s)}{R(s)} = \frac{496s + 2192}{s^4 + 18s^3 + 101s^2 + 208s + 320}$$

8.6   Optimal Regulator Design

The object of the optimal regulator design is to determine the optimal control law $\mathbf{u}^*(\mathbf{x}, t)$ which can transfer the system from its initial state to the final state (with zero system input) such that a given performance index is minimized. The performance index is selected to give the best trade-off between performance and cost of control. The performance index that is widely used in optimal control design is known as the *quadratic performance index* and is based on minimum-error and minimum-energy criteria.

Consider the plant described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{8.27}$$

The problem is to find the vector $\mathbf{K}$(t) of the control law

$$\mathbf{u}(t) = -\mathbf{K}(t)\mathbf{x}(t) \tag{8.28}$$

which minimizes the value of a quadratic performance index $\mathbf{J}$ of the form

$$\mathbf{J} = \int_{t_0}^{t_f} (\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u}) dt \tag{8.29}$$

subject to the dynamic plant equation in (8.27). In (8.29) $\mathbf{Q}$ is a positive semidefinite matrix and R is a real symmetric matrix. $\mathbf{Q}$ is positive semidefinite if all its principal minors are nonnegative. The choice of the elements of $\mathbf{Q}$ and $\mathbf{R}$ allows the relative weighting of individual state variables and individual control inputs.

To obtain a formal solution, we can use the method of *Lagrange multipliers*. The constraint problem is solved by augmenting (8.27) into (8.29) using an n-vector of *Lagrange multipliers*, λ. The problem reduces to the minimization of the following unconstrained function

$$\mathcal{L}(\mathbf{x}, \lambda, u, t) = [\mathbf{x}'\mathbf{Q}\mathbf{x} + \mathbf{u}'\mathbf{R}\mathbf{u}] + \lambda'[\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \dot{\mathbf{x}}] \tag{8.30}$$

The optimal values (denoted by the subscript $*$) are found by equating the partial derivatives to zero.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{A}\mathbf{X}^* + \mathbf{B}\mathbf{u}^* - \dot{\mathbf{x}}^* = \mathbf{0} \quad \Rightarrow \quad \dot{\mathbf{x}}^* = AX^* + Bu^* \tag{8.31}$$

$$\frac{\partial \mathcal{L}}{\partial u} = 2\mathbf{R}\mathbf{u}^* + \lambda'\mathbf{B} = \mathbf{0} \quad \Rightarrow \quad \mathbf{u}^* = -\frac{1}{2}\mathbf{R}^{-1}\lambda'\mathbf{B} \tag{8.32}$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2\mathbf{x}'^*\mathbf{Q} + \dot{\lambda}' + \lambda'\mathbf{A} = \mathbf{0} \quad \Rightarrow \quad \dot{\lambda} = -2\mathbf{Q}\mathbf{x}^* - \mathbf{A}'\lambda \tag{8.33}$$

Assume that there exists a symmetric, time varying positive definite matrix $\mathbf{p}$(t) satisfying

$$\lambda = 2\mathbf{p}(t)\mathbf{x}^* \tag{8.34}$$

Substituting (8.34) into (8.32) gives the optimal closed-loop control law

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{B}'\mathbf{p}(t)\mathbf{x}^* \tag{8.35}$$

Obtaining the derivative of (8.34), we have

$$\dot{\lambda} = 2(\dot{\mathbf{p}}\mathbf{x}^* + \mathbf{p}\dot{\mathbf{x}}^*) \tag{8.36}$$

Finally equating (8.33) with (8.36), we obtain

$$\dot{\mathbf{p}}(t) = -\mathbf{p}(t)A - \mathbf{A}'\mathbf{p}(t) - \mathbf{Q} + \mathbf{p}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{p}(t) \tag{8.37}$$

The above equation is referred to as the matrix *Riccati equation.* The boundary condition for (8.37) is $\mathbf{p}(t_f) = \mathbf{0}$. Therefore, (8.37) must be integrated backward in time. Since a numerical solution is performed forward in time, a dummy time variable $\tau = t_f - t$ is replaced for time t. Once the solution to (8.37) is obtained the solution of the state equation (8.31) in conjunction with the optimum control equation (8.35) is obtained.

The function $[\tau, \mathbf{p}, \mathbf{K}, \mathbf{t}, \mathbf{x}]$=**riccati** is developed for the time-domain solution of the Riccati equation. The function returns the solution of the matrix Riccati equation, $\mathbf{p}(\tau)$, the optimal feedback gain vector $\mathbf{k}(\tau)$, and the initial state response $\mathbf{x}(t)$. In order to use this function, the user must declare the function $[\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, \mathbf{t_0}, \mathbf{t_f}, \mathbf{x_0}]$=**system** $(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, \mathbf{t_0}, \mathbf{t_f}, \mathbf{x_0})$ containing system matrices and the performance index matrices in an M-file named **system.m**.

For linear time-invariant systems, since $\dot{\mathbf{p}} = \mathbf{0}$, when the process is of infinite duration, that is $t_f = \infty$, (8.37) reduces to the algebraic Riccati equation

$$\mathbf{p}A + \mathbf{A}'\mathbf{p} + \mathbf{Q} - \mathbf{p}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}'\mathbf{p} = \mathbf{0} \tag{8.38}$$

The *MATLAB Control System Toolbox* function **[k, p]=lqr2(A, B, Q, R)** can be used for the solution of the algebraic Riccati equation.

Example 8.5

Design a state feedback system for the plant described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

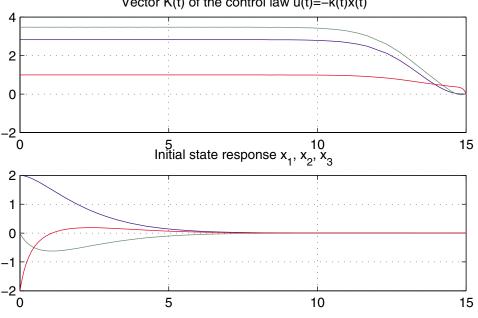$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Find the optimal control law to minimize the performance index

$$\mathbf{J} = \int_0^{10} \left(4x_1{}^2(t) + 3x_2{}^2 + 2x_3{}^2 + \frac{1}{2}u^2 \right) dt \tag{8.39}$$

The admissible states and control values are unconstrained. The states are initially at $x_1(0) = 2$, $x_2(0) = 0$ and $x_3(0) = -2$. For this system we have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad \text{and} \quad R = \frac{1}{2}$$

First an M-file named **system.m** is created and the function $[\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, t_0, t_f, x_0)$ = **system** $(\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{R}, t_0, t_f, x_0)$ is defined as follows.

FIGURE 8.4

Optimal feedback gain vector **K**(t) and initial condition response **x**(t).

```
function [A,B,Q,R,t0,tf,x0]=system(A,B,Q,R,t0,tf,x0)
A=[0 1 0; 0 0 1;0 -4 -5]; B=[0;0; 1];
Q=[4 0 0; 0 3 0;0 0 2];   R=.5;
t0=0; tf=15;
x0=[2 0 -2];
```

The above function is saved in an M-file named **system.m**. Then, the following com-
mands

```
[tt,p,k,t,x]=riccati
subplot(2,1,1), plot(tt,k),
title('Vector K(t) of the control law u(t)=-k(t)x(t)'),
grid,
subplot(2,1,2), plot(t,x),
title('Initial state response x_1, x_2, x_3'), grid
subplot(111)
```

result in the control law and the response is given in Figure 8.4.

Example 8.6

For Example 8.5 use the *MATLAB Control System Toolbox* **lqr2** to obtain the solution
to the algebraic Riccati equation.

The following commands

```
A=[0 1 0; 0 0 1;0 -4 -5]; B=[0;0; 1];
Q=[4 0 0; 0 3 0;0 0 2];R=.5;
[K, p]=lqr2(A,B,Q,R)
```

result in

```
    K =
        2.8284      3.4780      0.9963
    p =
       10.5755      8.4801      1.4142
        8.4801     11.0060      1.7390
        1.4142      1.7390      0.4982
```