

## Complex Numbers Review for EE-201 (Hadi saadat)

In our numbering system, *positive* numbers correspond to distance measured along a horizontal line to the right. *Negative* numbers are represented to the left of origin. Numbers corresponding to distances along the line shown in Figure 1 are called real numbers and the horizontal axis is known as the real axis.

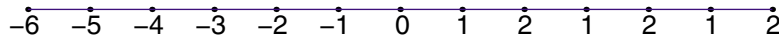


Figure 1. The system of real numbers.

Consider a point  $c$  in a two-dimensional plane located a distance  $M$  along a line at an angle  $\theta$ , taken counterclockwise from the positive horizontal axis.

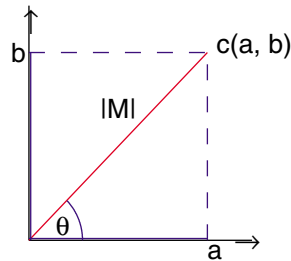


Figure 2. Graphical representation of a point in the  $x$ - $y$  plane (known as complex plane).

The projection of  $c$  along the horizontal axis or the so called real axis is shown by  $a$ . This component is known as the real component of  $c$ . The projection on the vertical axis is shown by  $b$ . Thus, we may show  $c$  by its two coordinates as  $c(a, b)$ . To differentiate between the two components, it is unfortunately common practice to call the vertical component of  $c$  the imaginary component. In this context, the vertical axis is known as the imaginary axis.

## Operator $j$

Consider a circle of radius unity with center at the origin of the  $x$ - $y$  plane as shown in Figure 3.

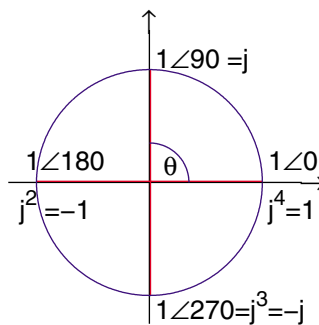


Figure 3. Graphical representation of the operator  $j$ .

The radius along the positive real axis may be written as  $1\angle 0$ . If this line of unity length is rotated counterclockwise by  $90^\circ$ , it will be located along the vertical axis, i.e.,  $1\angle 90^\circ$ . This operation is represented by the operator  $j$ , and we write

$$j = 1\angle 90^\circ$$

This means that if any real number is multiplied by  $j$  it is rotated by  $90^\circ$  in a counterclockwise direction.

Rotating through another 90, we have

$$(j)(j) = (1\angle 90^\circ)(1\angle 90^\circ) = 1\angle 180^\circ = -1$$

i.e.,

$$j^2 = -1$$

Taking the square root, we may write

$$j = \sqrt{-1}$$

The equation  $j^2 = -1$  is a statement that the two *operations* are equivalent. However, and that is the beauty of the concept, in all algebraic computation imaginary numbers can be handled as if  $j$  had a numerical value.

Continuing the rotation by another  $90^\circ$ , i.e., a total of  $270^\circ$ , we write

$$j^3 = (j^2)(j) = -j = 1\angle 270^\circ = 1\angle -90^\circ$$

and

$$j^4 = (j^2)(j^2) = (-1)(-1) = 1\angle 360^\circ = 1\angle 0^\circ$$

We can write the reciprocal operation  $\frac{1}{j}$  as

$$\frac{1}{j} = \frac{(j)}{(j)(j)} = \frac{j}{j^2} = \frac{j}{-1} = -j = 1\angle -90^\circ$$

## Rectangular and Polar Forms

With the introduction of the operator  $j$ , the point  $c(a, b)$  in Figure 2 may be represented as

$$c = a + jb$$

This representation indicates that the real part,  $a$ , is measured along the real axis (the abscissa) and the so called imaginary part,  $b$ , is reckoned along the imaginary axis (the ordinate). This representation is known as the *rectangular form* of a complex number  $c$ .

The complex number  $c = a + jb$  may also be represented as the length or magnitude of a line segment,  $|M|$ , at an angle,  $\theta$ , as indicated in Figure 2. Thus,

$$c = a + jb = |M|\angle\theta$$

The form  $c = |M|\angle\theta$  is called the *polar form*,  $|M|$  is the magnitude and  $\theta$  is called the *angle* or the *argument* of  $c$ . The conversion from rectangular to polar form can be deduced immediately from Figure 2 in conjunction with the Pythagorean theorem, i.e.,

$$|M| = \sqrt{a^2 + b^2}$$

and the angle is given by

$$\theta = \tan^{-1} \frac{b}{a}$$

For conversion from polar to rectangular from,

$$a = |M| \cos \theta$$

$$b = |M| \sin \theta$$

Thus,  $c$  can be written as

$$c = |M|(\cos \theta + j \sin \theta)$$

## The Euler's identity – Exponential Form

Consider a complex number with the Magnitude  $|M| = 1$ ,

$$c = 1\angle\theta = \cos\theta + j\sin\theta$$

Taking derivative of  $c$  with respect to  $\theta$ , result in

$$\frac{dc}{d\theta} = -\sin\theta + j\cos\theta = j(\cos\theta + j\sin\theta)$$

or

$$\frac{dc}{d\theta} = jc$$

Separating the variable,

$$\frac{1}{c}dc = jd\theta$$

Integrating the above equation, we get

$$\ln c = j\theta + K$$

where  $K$  is the constant of integration. The magnitude of  $c$  is unity regardless of the angle, therefore, at  $\theta = 0$ ,  $\ln(1) = j(0) + K$ , or  $K = 0$ , and

$$\ln c = j\theta$$

or

$$c = e^{j\theta}$$

With  $c$  given by the equation  $c = \cos\theta + j\sin\theta$ , the *exponential form* also known as the *Euler's identity* is

$$e^{j\theta} = \cos\theta + j\sin\theta$$

for an angle  $-\theta$ , we obtain

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Adding and subtracting the above two equations lead to the representation for  $\cos\theta$  and  $\sin\theta$ ,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

With the addition of the Euler's identity, the three way of representing a complex number, rectangular, polar and exponential forms are all equivalent and we may write

$$c = a + jb = |M|\angle\theta = |M|e^{j\theta}$$

## Mathematical Operations

The conjugate of a complex number  $c = a + jb$  denoted by  $c^*$  and is defined as

$$c^* = a - jb$$

or in polar form for  $c = |M|\angle\theta$ , we have

$$c^* = |M|\angle-\theta$$

To add or subtract two complex numbers, we add (or subtract) their real parts and their imaginary parts. For two complex numbers designated by  $c_1 = a_1 + jb_1$  and  $c_2 = a_2 + jb_2$ , their sum is

$$c_1 + c_2 = (a_1 + a_2) + j(b_1 + b_2)$$

The multiplication of  $c_1$  and  $c_2$  in rectangular form is obtained as follows (note  $j^2 = -1$ )

$$c_1 c_2 = (a_1 + jb_1)(a_2 + jb_2) = a_1 a_2 - b_1 b_2 + j(a_1 b_2 + b_1 a_2)$$

or in polar form for  $c_1 = |M_1| \angle \theta_1$ , and  $c_2 = |M_2| \angle \theta_2$ , The multiplication of  $c_1$  and  $c_2$  is

$$\begin{aligned} c_1 c_2 &= (|M_1| \angle \theta_1)(|M_2| \angle \theta_2) \\ &= (|M_1| e^{j\theta_1})(|M_2| e^{j\theta_2}) \\ &= |M_1| |M_2| e^{j(\theta_1 + \theta_2)} \\ &= |M_1| |M_2| \angle \theta_1 + \theta_2 \end{aligned}$$

if a complex number  $c = a + jb$  is multiplied by its conjugate  $c^* = a - jb$ , the result is

$$cc^* = (a + jb)(a - jb) = a^2 + b^2 = |M|^2$$

or in polar form

$$cc^* = |M| \angle \theta |M| \angle -\theta = |M|^2$$

For the division of  $c_1$  by  $c_2$  in rectangular form, we multiply numerator and denominator by the conjugate of the denominator, this results in

$$\begin{aligned} \frac{c_1}{c_2} &= \frac{(a_1 + jb_1)}{(a_2 + jb_2)} \\ &= \frac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)} \\ &= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + j \frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2} \end{aligned}$$

or in polar form for  $c_1 = |M_1| \angle \theta_1$ , and  $c_2 = |M_2| \angle \theta_2$ , the division of  $c_1$  by  $c_2$  is

$$\begin{aligned} \frac{c_1}{c_2} &= \frac{|M_1| \angle \theta_1}{|M_2| \angle \theta_2} \\ &= \frac{|M_1| e^{j\theta_1}}{|M_2| e^{j\theta_2}} \\ &= \frac{|M_1|}{|M_2|} e^{j(\theta_1 - \theta_2)} \\ &= \frac{|M_1|}{|M_2|} \angle \theta_1 - \theta_2 \end{aligned}$$

### Example 1

For the complex numbers

$$c_1 = 20 \angle 36.87^\circ, \quad c_2 = 40 \angle -53.13^\circ, \quad c_3 = 40 + j80, \quad c_4 = 12 \angle \frac{\pi}{6}, \quad c_5 = 5 \angle -\frac{\pi}{6}, \quad \text{and} \quad c_6 = 30 + j40$$

Find  $c = (c_1 + c_2 + c_3) / (c_4 * c_5 - c_6)$

Substituting for the values and converting  $c_1$  and  $c_2$  to rectangular form, we have

$$\begin{aligned} c &= \frac{20 \angle 36.87^\circ + 40 \angle -53.13^\circ + 40 + j80}{(12 \angle \frac{\pi}{6})(5 \angle -\frac{\pi}{6}) - (30 + j40)} \\ &= \frac{(16 + j12) + (24 - j32) + (40 + j80)}{(60 + j0) - (30 + j40)} = \frac{80 + j60}{30 - j40} \\ &= \frac{100 \angle 36.87^\circ}{50 \angle -53.13^\circ} = 2 \angle 90^\circ = j2 \end{aligned}$$

Use MATLAB To evaluate the complex number  $c$  described in Example 1. In MATLAB, we use the following commands:

```
c1 = 20*exp(j*36.87*pi/180); % In MATLAB angles must be in radian
c2 = 40*exp(-j*53.13*pi/180);
c3 = 40 + j*80;
c4 = 12*exp(j*pi/6);
c5 = 5*exp(-j*pi/6);
c6 = 30 + j*40;
c = (c1+c2+c3)/(c4*c5-c6)
M=abs(c) % Magnitude
theta = angle(c)*180/pi % Angle in degree
```

Save in a file with extension m, and run to get the result

```
c =
    0.0000 + 2.0000i
M =
    2.0000
theta =
    90.0000
```

## Example 2

A complex function is described by

$$g = \frac{2500(j\omega)}{(25 + j\omega)(100 + j\omega)}$$

Write an script m-file to evaluate the magnitude and phase angle of  $g$  for  $\omega$  from 0 to 200 in step of 1, and obtain the magnitude and phase angle plots versus  $\omega$ .

We use the following statements:

```
w=0:1:200;
g= (2500*j*w)./((25+j*w).*(100+j*w)); %Array Multiplication & division use .* & ./
M=abs(g); % Magnitude
theta = angle(g)*180/pi; % Angle in degree
subplot(2,1,1), plot(w, M), grid
ylabel('M'), xlabel('\omega')
subplot(2,1,2), plot(w, theta), grid
ylabel('\theta, degree'), xlabel('\omega')
```

The result is

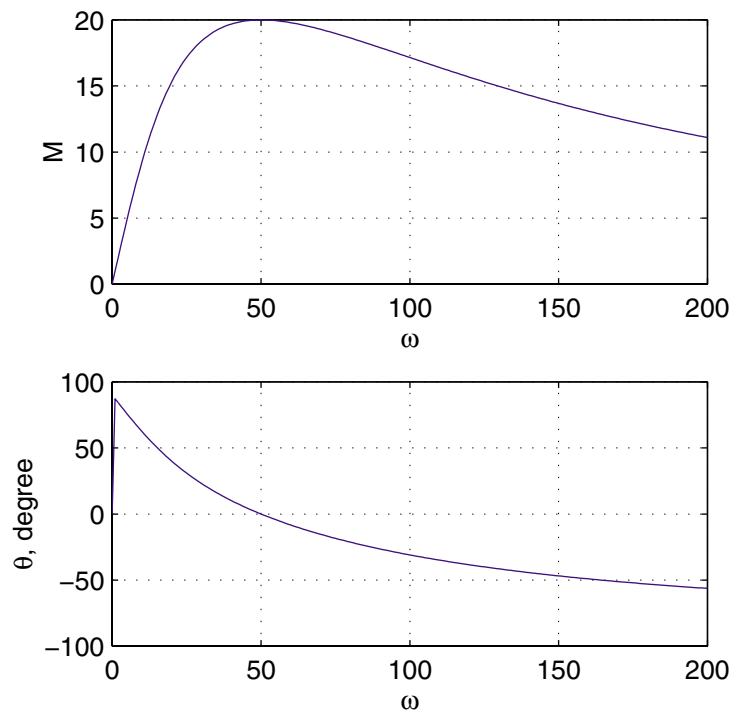


Figure 4. Magnitude and phase angle plots for complex function in Example 2.

### Homework Problems

1. Using your calculator evaluate  $g$  for the function in Example 2 for (a)  $\omega = 11$ , (b)  $\omega = 50$ , and (c)  $\omega = 112$ . Express your answers both in rectangular and polar forms.

2. A complex function is described by

$$g = \frac{10000}{(10 + j\omega)(100 - \omega^2 + 10(j\omega))}$$

Write an script m-file to evaluate the magnitude and phase angle of  $g$  for  $\omega$  from 0 to 50 in step of 1, and obtain the magnitude and phase angle plots versus  $\omega$ .

3. Using your calculator evaluate  $g$  for the function in Problem 2 for (a)  $\omega = 10$ , (b)  $\omega = 14.142$ , and (c)  $\omega = 21.5$ . Express your answers both in rectangular and polar forms.